

# DIAGONAL PROPERTY OF THE SYMMETRIC PRODUCT OF A SMOOTH CURVE

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**ABSTRACT.** Let  $C$  be an irreducible smooth projective curve defined over an algebraically closed field. We prove that the symmetric product  $\mathrm{Sym}^d(C)$  has the diagonal property for all  $d \geq 1$ . For any positive integers  $n$  and  $r$ , let  $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(nr)$  be the Quot scheme parametrizing all the torsion quotients of  $\mathcal{O}_C^{\oplus n}$  of degree  $nr$ . We prove that  $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(nr)$  has the weak point property.

## 1. INTRODUCTION

In [PSP], Pragacz, Srinivas and Pati introduced the diagonal and (weak) point properties of a variety, which we recall.

Let  $X$  be a variety of dimension  $d$  over an algebraically closed field  $k$ . It is said to have the *diagonal property* if there is a vector bundle  $E \rightarrow X \times X$  of rank  $d$ , and a section  $s \in H^0(X \times X, E)$ , such that the zero scheme of  $s$  is the diagonal in  $X \times X$ . The variety  $X$  is said to have the *weak point property* if there is a vector bundle  $F$  on  $X$  of rank  $d$ , and a section  $t \in H^0(X, F)$ , such that the zero scheme of  $t$  is a (reduced) point of  $X$ . The diagonal property implies the weak point property because the restriction of the above section  $s$  to  $X \times \{x_0\}$  vanishes exactly on  $x_0$ .

These properties were extensively studied in [PSP] and [De]. In particular, it was shown that

- they impose strong conditions on the variety,
- on the other hand there are many example of varieties with these properties.

Here we investigate these conditions for some varieties associated to a smooth projective curve.

Let  $C$  be an irreducible smooth projective curve over  $k$ . For any positive integer  $d$ , let  $\mathrm{Sym}^d(C)$  be the quotient of  $C^d$  for the natural action of the group of permutations of  $\{1, \dots, d\}$ . It is a smooth projective variety of dimension  $d$ . We prove the following (Theorem 3.1):

**Theorem 1.1.** *The variety  $\mathrm{Sym}^d(C)$  has the diagonal property.*

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Theorem 1 in [PSP, p. 1236] contains several examples of surfaces satisfying the diagonal property. We note that the surface  $\text{Sym}^2(C)$  is not among them.

For positive integers  $n$  and  $d$ , let  $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$  be the Quot scheme parametrizing the torsion quotients of  $\mathcal{O}_C^{\oplus n}$  of degree  $d$ . Quot schemes were constructed in [Gr] (see [Ni] for an exposition on [Gr]). The variety  $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$  is smooth projective, and its dimension is  $nd$ . Note that  $\mathcal{Q}_{\mathcal{O}_C}(d) = \text{Sym}^d(C)$ . These varieties  $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$  are extensively studied in algebraic geometry and mathematical physics (see [BGL], [BDW], [Ba], [BR] and references therein).

We prove the following (Theorem 2.2):

**Theorem 1.2.** *If  $d$  is a multiple of  $n$ , then the variety  $\mathcal{Q}_{\mathcal{O}_C^{\oplus n}}(d)$  has the weak point property.*

## 2. QUOT SCHEME AND THE WEAK POINT PROPERTY

We continue with the notation of the introduction.

For a locally free coherent sheaf  $E$  of rank  $n$  on  $C$ , let  $\mathcal{Q}_E(d)$  be the Quot scheme parametrizing all torsion quotients of  $E$  of degree  $d$ . Equivalently,  $\mathcal{Q}_E(d)$  parametrizes all coherent subsheaves of  $E$  of rank  $n$  and degree  $\text{degree}(E) - d$ . Note that any coherent subsheaf of  $E$  is locally free because any torsionfree coherent sheaf on a smooth curve is locally free. This  $\mathcal{Q}_E(d)$  is an irreducible smooth projective variety of dimension  $nd$ .

There is a natural morphism

$$\varphi' : \mathcal{Q}_E(d) \longrightarrow \mathcal{Q}_{\wedge^n E}(d)$$

that sends any subsheaf  $S \subset E$  of rank  $n$  and degree  $\text{degree}(E) - d$  to the subsheaf  $\wedge^n S \subset \wedge^n E$ . Next note that  $\mathcal{Q}_{\wedge^n E}(d)$  is identified with the symmetric product  $\text{Sym}^d(C)$  by sending any subsheaf  $S' \subset \wedge^n E$  to the scheme theoretic support of the quotient sheaf  $(\wedge^n E)/S'$ . Let

$$(2.1) \quad \varphi : \mathcal{Q}_E(d) \longrightarrow \text{Sym}^d(C)$$

be the composition of  $\varphi'$  with this identification of  $\mathcal{Q}_{\wedge^n E}(d)$  with  $\text{Sym}^d(C)$ . It should be mentioned that for a subsheaf  $S \subset E$  of rank  $n$  and degree  $\text{degree}(E) - d$ , the image  $\varphi(S) \in \text{Sym}^d(C)$  does not, in general, coincide with the scheme theoretic support of the quotient sheaf  $E/S$ .

The symmetric product  $\text{Sym}^d(C)$  is the moduli space of effective divisors of degree  $d$  on  $C$ . Let

$$(2.2) \quad D \subset Y := C \times \text{Sym}^d(C)$$

be the universal divisor. So the fiber of  $D$  over a point  $a \in \text{Sym}^d(C)$  is the zero dimensional subscheme of  $C$  of length  $d$  defined by  $a$ . Let

$$(2.3) \quad \mathcal{D} = (\text{Id}_C \times \varphi)^{-1}(D) \subset C \times \mathcal{Q}_E(d)$$

be the inverse image of  $D$ , where  $\varphi$  is constructed in (2.1).

**Remark 2.1.** Let  $L$  be a line bundle on  $C$ . For  $E$  as above, if  $S \subset E$  is a subsheaf of rank  $n$  and degree  $\deg(E) - d$ , then

$$S \otimes L \subset E \otimes L$$

is a subsheaf of rank  $n$  and degree  $\deg(E \otimes L) - d$ . Therefore, we get an isomorphism

$$\mathcal{Q}_E(d) \xrightarrow{\sim} \mathcal{Q}_{E \otimes L}(d)$$

by sending any subsheaf  $S \subset E$  to the subsheaf  $S \otimes L \subset E \otimes L$ .

**Theorem 2.2.** *For positive integers  $d, n$  such that  $d$  is a multiple of  $n$ , the Quot scheme  $\mathcal{Q}_{\mathcal{O}_C^n}(d)$  satisfies the weak point property.*

*Proof.* Let  $r \in \mathbb{N}$  be such that  $d = rn$ . Fix a closed point  $x_0$  in  $C$ . The line bundle  $\mathcal{O}_C(rx_0)$  on  $C$  will be denoted by  $L$ . By Remark 2.1 it is enough to prove the weak point property for  $\mathcal{Q}_{L^{\oplus n}}(d)$ .

Let  $\mathcal{D} \hookrightarrow C \times \mathcal{Q}_{L^{\oplus n}}(d)$  be the divisor constructed in (2.3). Let

$$(2.4) \quad p : \mathcal{D} \longrightarrow C \quad \text{and} \quad q : \mathcal{D} \longrightarrow \mathcal{Q}_{L^{\oplus n}}(d)$$

be the projections. Taking the direct sum of copies of the natural inclusion

$$\iota : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(rx_0),$$

we get a short exact sequence of sheaves on  $C$

$$(2.5) \quad 0 \longrightarrow \mathcal{O}_C^{\oplus n} \xrightarrow{\iota^{\oplus n}} \mathcal{O}_C(rx_0)^{\oplus n} \longrightarrow T \longrightarrow 0,$$

where  $T$  is a torsion sheaf on  $C$  of degree  $nr = d$ . Therefore, this quotient  $T$  is represented by a point of  $\mathcal{Q}_{L^{\oplus n}}(d)$ . Let

$$(2.6) \quad t_0 \in \mathcal{Q}_{L^{\oplus n}}(d)$$

be the point representing  $T$ .

The direct image

$$F := q_* p^* L^{\oplus n} \longrightarrow \mathcal{Q}_{L^{\oplus n}}(d)$$

is a vector bundle of rank  $nd$ , where  $p$  and  $q$  are the projections in (2.4). We will construct a section of  $F$ . The section of  $L = \mathcal{O}_C(rx_0)$  given by the constant function 1 will be denoted by  $s_0$ . Consider the section

$$s := \iota^{\oplus n}(s_0^{\oplus n}) \in H^0(C, L^{\oplus n}),$$

where  $\iota^{\oplus n}$  is the homomorphism in (2.5). We have

$$(2.7) \quad \tilde{s} := q_* p^* s \in H^0(\mathcal{Q}_{L^{\oplus n}}(d), F).$$

For the point  $t_0$  in (2.6), the scheme theoretic inverse image

$$q^{-1}(t_0) \subset \mathcal{D} \subset C \times \mathcal{Q}_{L^{\oplus n}}(d)$$

is  $(rx_0) \times t_0$ , where  $q$  is the projection in (2.4). Since the section  $s_0$  vanishes exactly on  $rx_0$ , this implies that the section  $\tilde{s}$  in (2.7) vanishes exactly on the reduced point  $t_0$ . Therefore,  $\mathcal{Q}_{L^{\oplus n}}(d)$  has the weak point property.  $\square$

## 3. DIAGONAL PROPERTY FOR SYMMETRIC PRODUCT OF CURVES

**Theorem 3.1.** *For any  $d \geq 1$ , the symmetric product  $\text{Sym}^d(C)$  of a smooth projective curve  $C$  has the diagonal property.*

*Proof.* Consider the divisor  $D$  in (2.2). Let

$$(3.1) \quad L = \mathcal{O}_Y(D) \longrightarrow Y$$

be the line bundle. Now consider  $Z := Y \times \text{Sym}^d(C) = C \times \text{Sym}^d(C) \times \text{Sym}^d(C)$ . Let

$$(3.2) \quad \alpha : Z \longrightarrow C, \beta : Z \longrightarrow \text{Sym}^d(C) \quad \text{and} \quad \gamma : Z \longrightarrow \text{Sym}^d(C)$$

be the projections defined by  $(x, y, z) \longmapsto x$ ,  $(x, y, z) \longmapsto y$  and  $(x, y, z) \longmapsto z$  respectively. Let

$$(3.3) \quad \tilde{D} := (\alpha \times \gamma)^{-1}(D) \subset C \times \text{Sym}^d(C) \times \text{Sym}^d(C) = Z$$

be the inverse image, where  $D$  is defined in (2.2).

Let

$$p : \tilde{D} \longrightarrow \text{Sym}^d(C) \times \text{Sym}^d(C)$$

be the projection defined by  $b \longmapsto (\beta(b), \gamma(b))$ , where  $\beta$  and  $\gamma$  are defined in (3.2), and  $\tilde{D}$  is constructed in (3.3). Consider the direct image

$$(3.4) \quad V := p_*((\alpha \times \beta)^* L|_{\tilde{D}}) \longrightarrow \text{Sym}^d(C) \times \text{Sym}^d(C),$$

where  $L$  is the line bundle in (3.1). The natural projection

$$D \longrightarrow \text{Sym}^d(C), \quad (x, y) \longmapsto y,$$

where  $D$  is defined in (2.2), is a finite morphism of degree  $d$ . This implies that  $p$  is a finite morphism of degree  $d$ . Consequently, the direct image  $V$  is a vector bundle on  $\text{Sym}^d(C) \times \text{Sym}^d(C)$  of rank  $d$ .

Consider the natural inclusion  $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(D) = L$  (see (3.1)). Let

$$(3.5) \quad \sigma_0 \in H^0(Y, L)$$

be the section given by the constant function 1 using this inclusion. Let

$$\sigma := p_*(((\alpha \times \beta)^* \sigma_0)|_{\tilde{D}}) \in H^0(\text{Sym}^d(C) \times \text{Sym}^d(C), V)$$

be the section of  $V$  (constructed in (3.4)) given by  $\sigma_0$ .

We will show that the scheme theoretic inverse image

$$\sigma^{-1}(0) \subset \text{Sym}^d(C) \times \text{Sym}^d(C)$$

is the diagonal.

Take any point  $(a, b) \in \text{Sym}^d(C) \times \text{Sym}^d(C)$  such that  $a \neq b$ . Then there is a point  $z \in C$  such that the multiplicity of  $z$  in  $a$  is strictly smaller than the multiplicity of  $z$  in  $b$ . We note that the scheme theoretic inverse image

$$p^{-1}((a, b)) \subset \tilde{D} \subset Z = C \times \text{Sym}^d(C) \times \text{Sym}^d(C)$$

is  $\{(a, b)\} \times \widehat{b}$ , where  $\widehat{b}$  is the zero dimensional subscheme of  $C$  of length  $d$  defined by  $b$ . On the other hand, for the section  $\sigma_0$  in (3.5), the intersection  $\sigma_0^{-1}(0) \cap (C \times \{a\})$  is the zero dimensional subscheme  $\widehat{a}$  of  $C$  of length  $d$  defined by  $a$ . Since the multiplicity of  $z$  in  $a$  is strictly smaller than the multiplicity of  $z$  in  $b$ , we have

$$\sigma_0((z_0, b)) \neq 0.$$

Consequently,  $\sigma((a, b)) \neq 0$ .

Now take a point  $(a, a)$  on the diagonal of  $\text{Sym}^d(C) \times \text{Sym}^d(C)$ . We have observed above that the inverse image

$$p^{-1}((a, a)) \subset C$$

coincides with the intersection  $\sigma_0^{-1}(0) \cap (C \times a)$ . This implies that

- $\sigma((a, a)) = 0$ , and
- $\sigma^{-1}(0)$  is the reduced diagonal.

Therefore,  $\text{Sym}^d(C)$  has the diagonal property. □

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